INTRODUCTION TO LINEAR ALGEBRA

31.10.2016,
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Slides adapted from Math Tools for Neuroscience course presented 2016 by Lane McIntosh & Kiah Hardcastle (Stanford University)

Overview

- When and why is linear algebra useful?
- Vectors and their operations
- Matrices and their operations
- Special matrices
- Determinants
- Eigenvalues and eigenvectors

Why linear algebra?

1.63	5.20	7.66	8.12	3.22
4.98	5.90	8.21	9.29	20.10
10.10	8.57	5.73	8.17	2.22
0.02	0.21	0.14	0.93	1.40
9.27	10.27	13.12	8.90	9.01
7.44	6.98	5.62	8.20	7.21
100.10	8.22	7.54	60.10	1.69
40.20	29.21	12.45	10.41	8.90
32.33	21.59	10.21	4.99	2.62
2.99	1.67	1.01	0.80	0.07

Datasets are matrices

	time ⟶				
neuron 1	1.63	5.20	7.66	8.12	3.22
neuron 2	4.98	5.90	8.21	9.29	20.10
neuron 3	10.10	8.57	5.73	8.17	2.22
neuron 4	0.02	0.21	0.14	0.93	1.40
neuron 5	9.27	10.27	13.12	8.90	9.01
neuron 6	7.44	6.98	5.62	8.20	7.21
neuron 7	100.10	8.22	7.54	60.10	1.69
neuron 8	40.20	29.21	12.45	10.41	8.90
neuron 9	32.33	21.59	10.21	4.99	2.62
neuron 10	2.99	1.67	1.01	0.80	0.07

Datasets are matrices

time →

pixel 1	1.63	5.20	7.66	8.12	3.22
pixel 2	4.98	5.90	8.21	9.29	20.10
pixel 3	10.10	8.57	5.73	8.17	2.22
pixel 4	0.02	0.21	0.14	0.93	1.40
pixel 5	9.27	10.27	13.12	8.90	9.01
pixel 6	7.44	6.98	5.62	8.20	7.21
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pixel 9	32.33	21.59	10.21	4.99	2.62
pixel 10	2.99	1.67	1.01	0.80	0.07

Datasets are matrices

patient →

gene 1	1.63	5.20	7.66	8.12	3.22
gene 2	4.98	5.90	8.21	9.29	20.10
gene 3	10.10	8.57	5.73	8.17	2.22
gene 4	0.02	0.21	0.14	0.93	1.40
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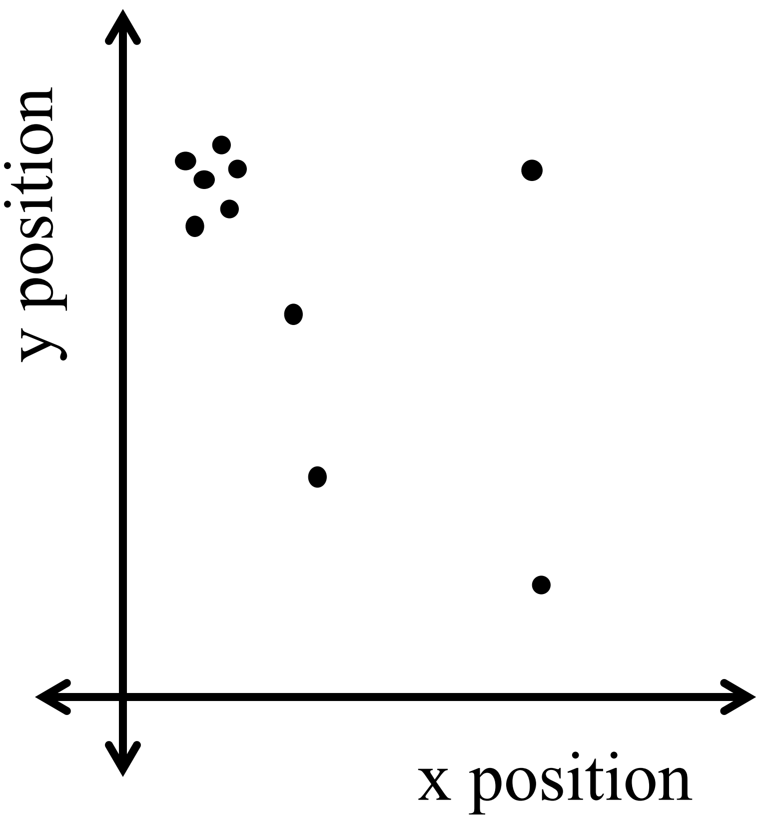
gene 9	32.33	21.59	10.21	4.99	2.62
gene 10	2.99	1.67	1.01	0.80	0.07

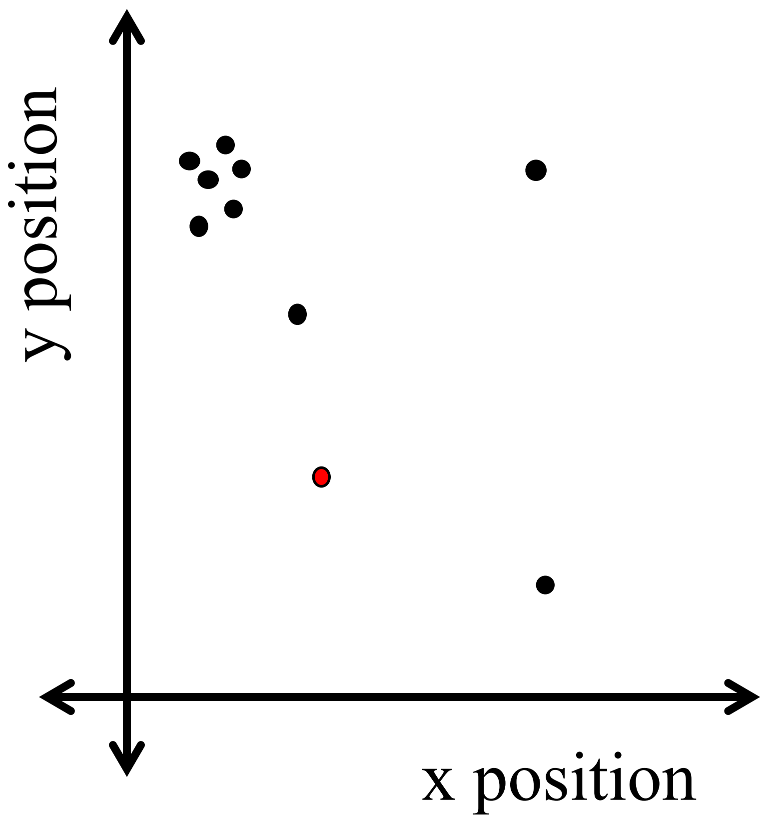
Matrix Arithmetic

Measure a mouse's location



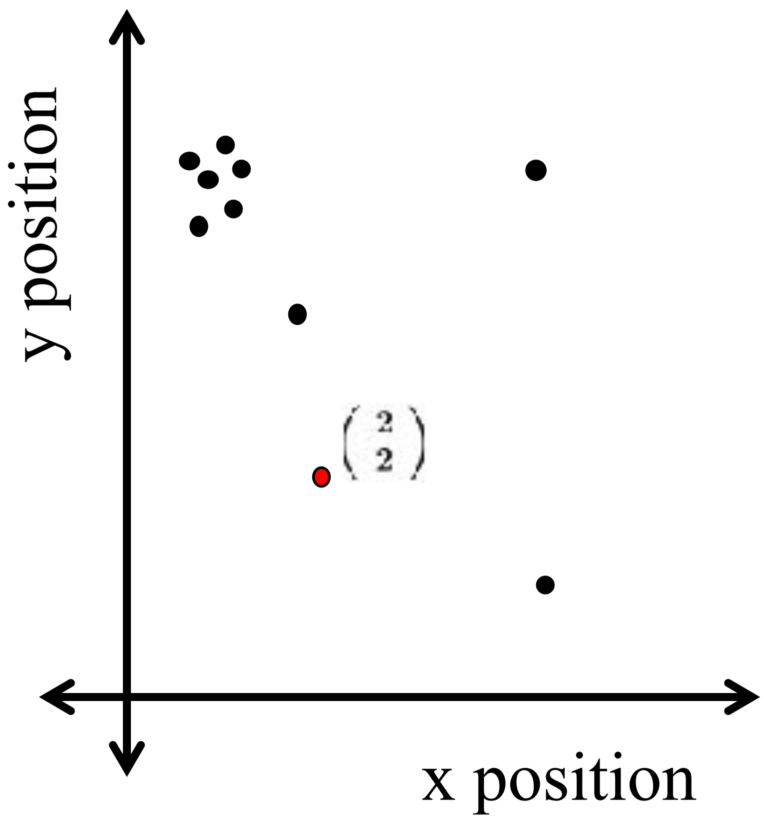
Matrix Arithmetic

Measure a mouse's location



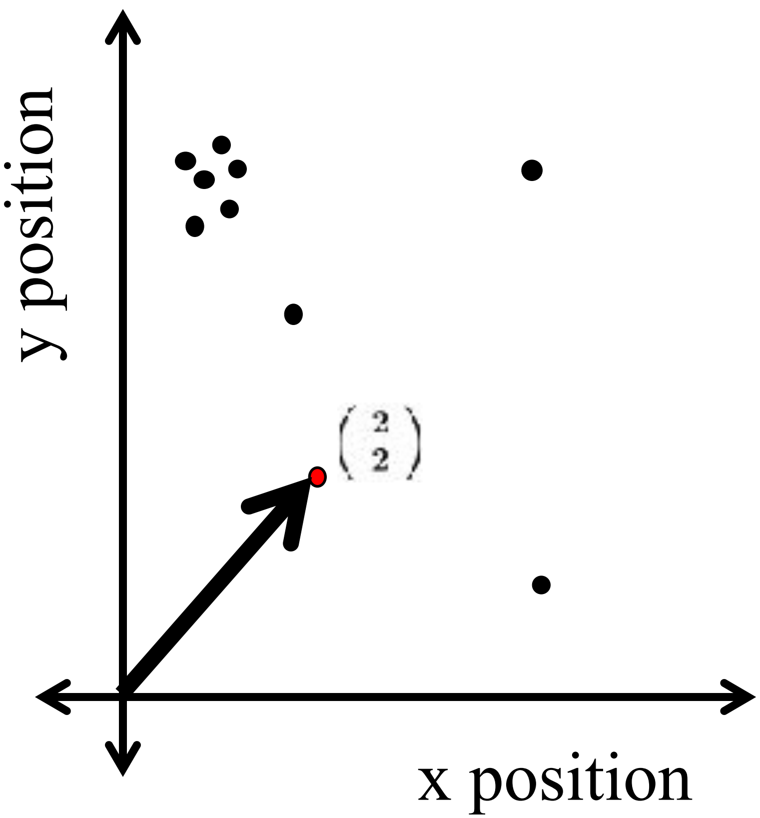
Matrix Arithmetic

Measure a mouse's location



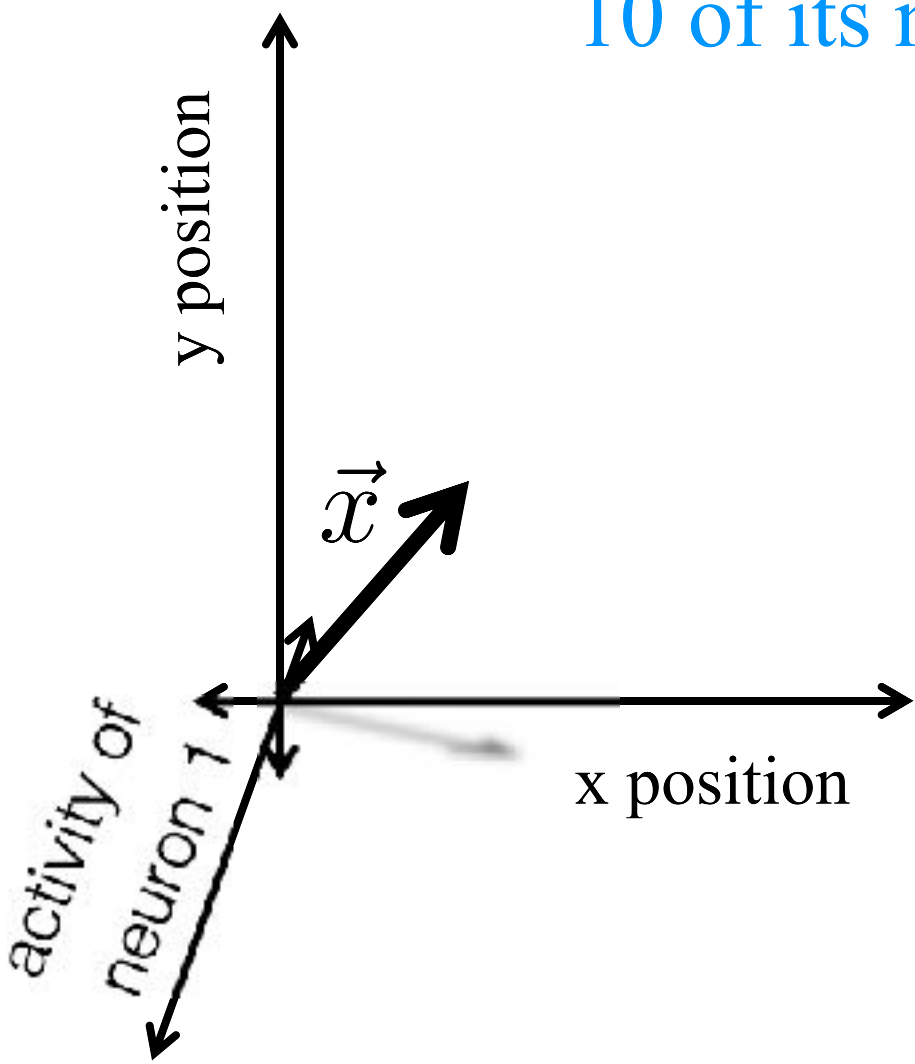
Matrix Arithmetic

Measure a mouse's location

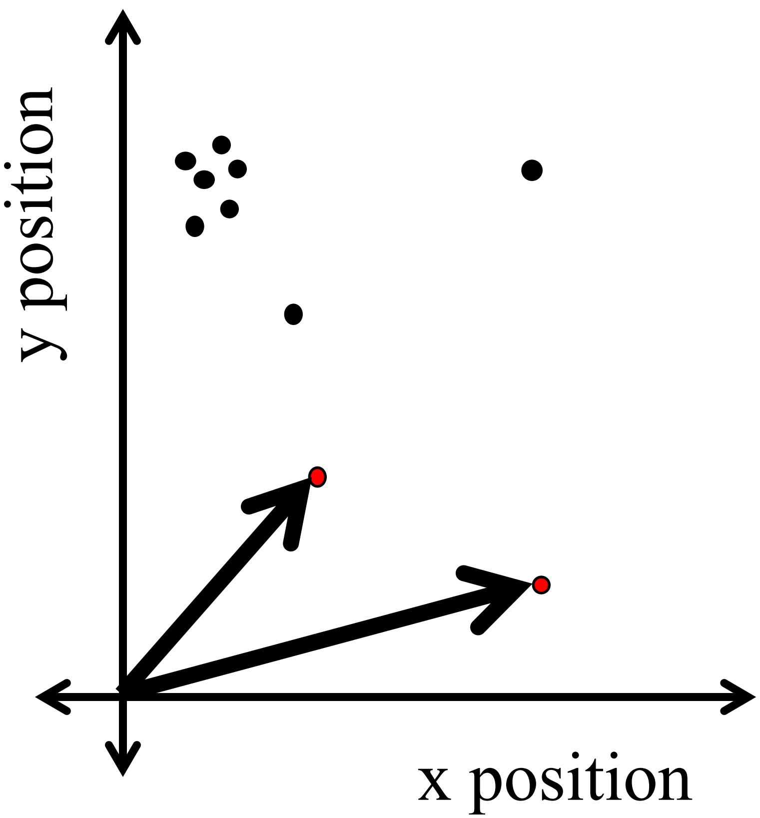


Matrix Arithmetic

Measure a mouse's location and 10 of its neurons

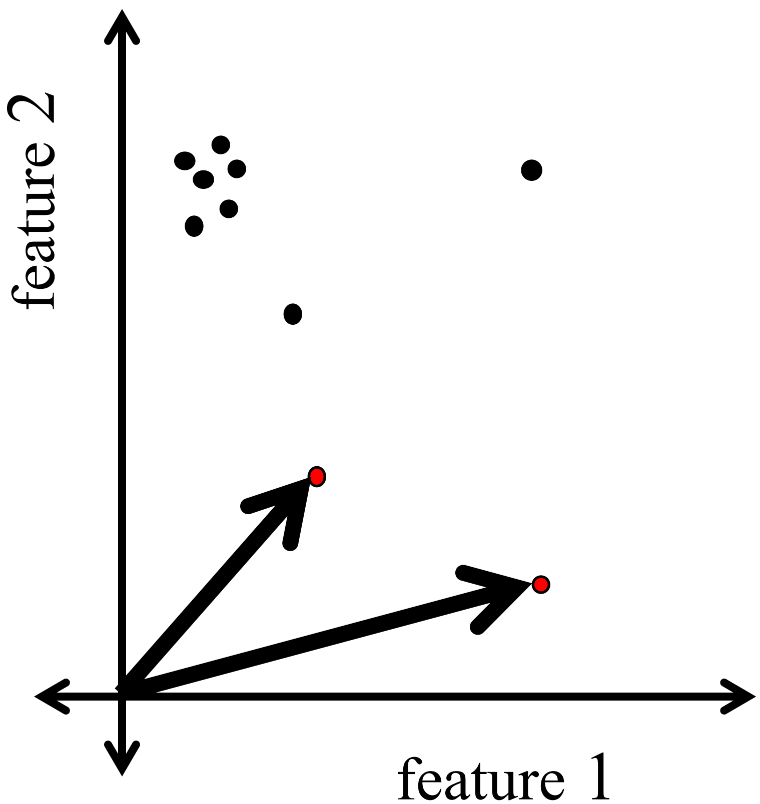


Matrix addition and subtraction



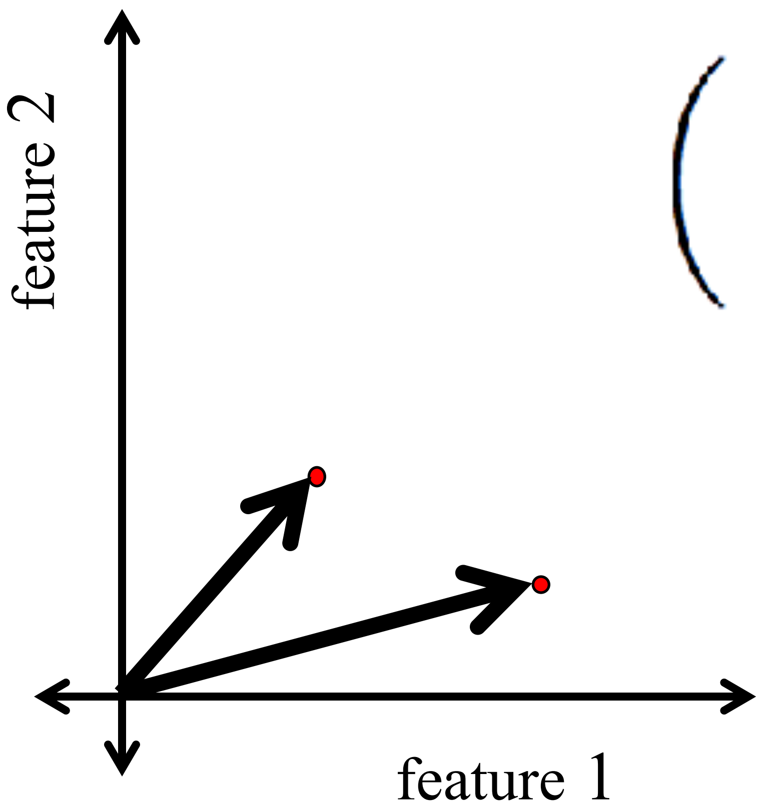
$$\begin{array}{l c} & \begin{array}{cc} \text{measurement 1} & \text{measurement 2} \end{array} \\ \begin{array}{l} \text{x position} \\ \text{y position} \end{array} & \begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix} \end{array}$$

Matrix addition and subtraction



$$
\begin{array}{l c}
 & \begin{array}{cc} \text{measurement 1} & \text{measurement 2} \end{array} \\
\begin{array}{l} \text{feature 1} \\ \text{feature 2} \end{array} & \begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix}
\end{array}
$$

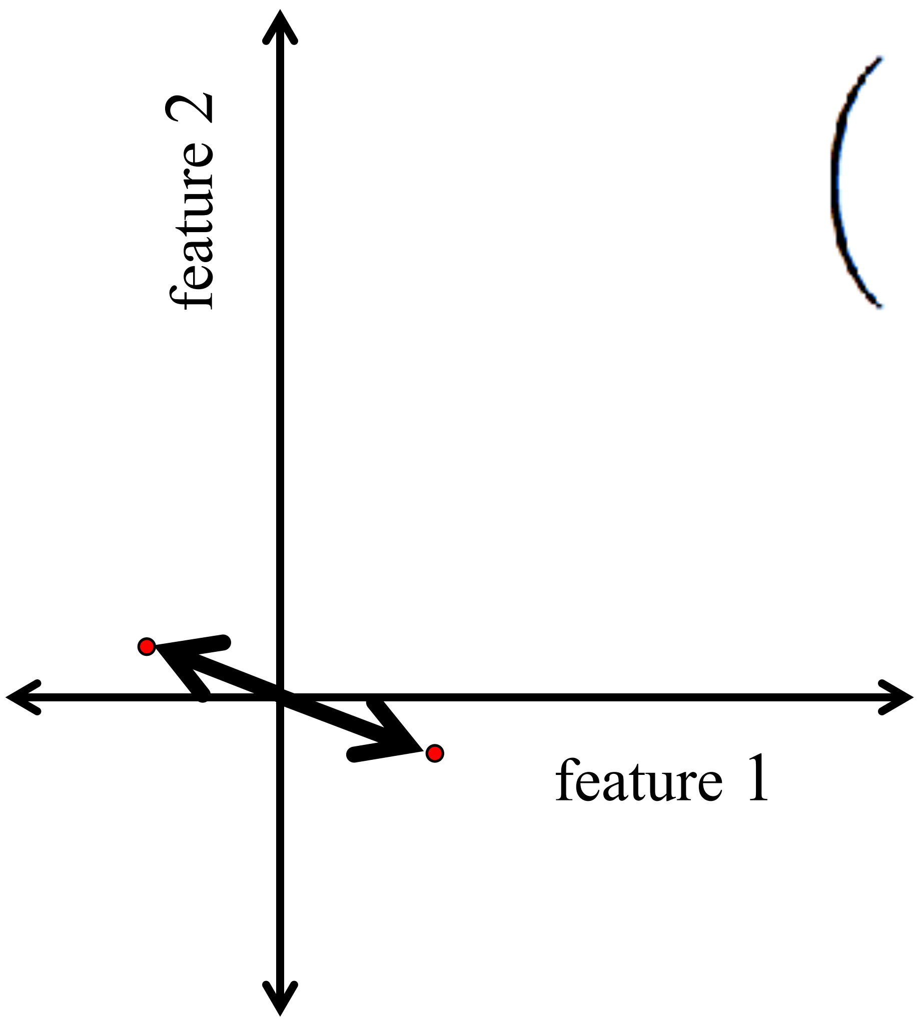
Matrix addition and subtraction



subtract the mean:

$$\begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1.5 \end{pmatrix}$$

Matrix addition and subtraction

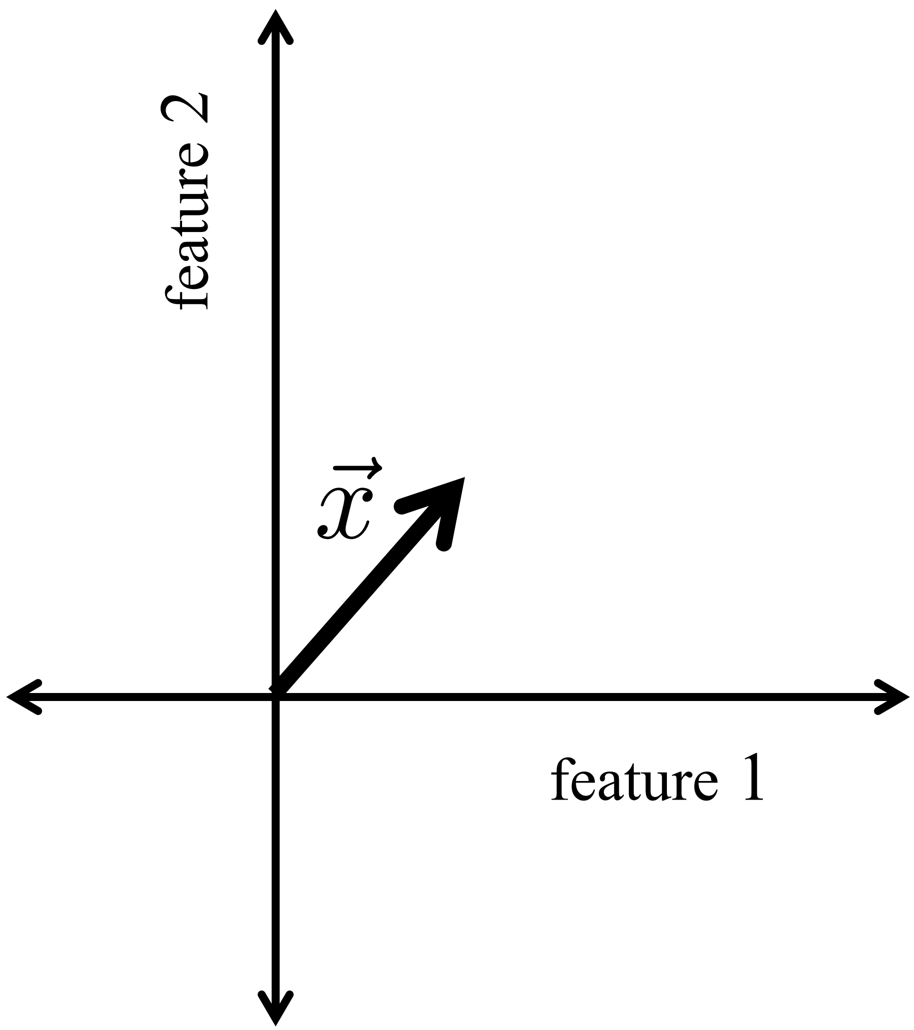


subtract the mean:

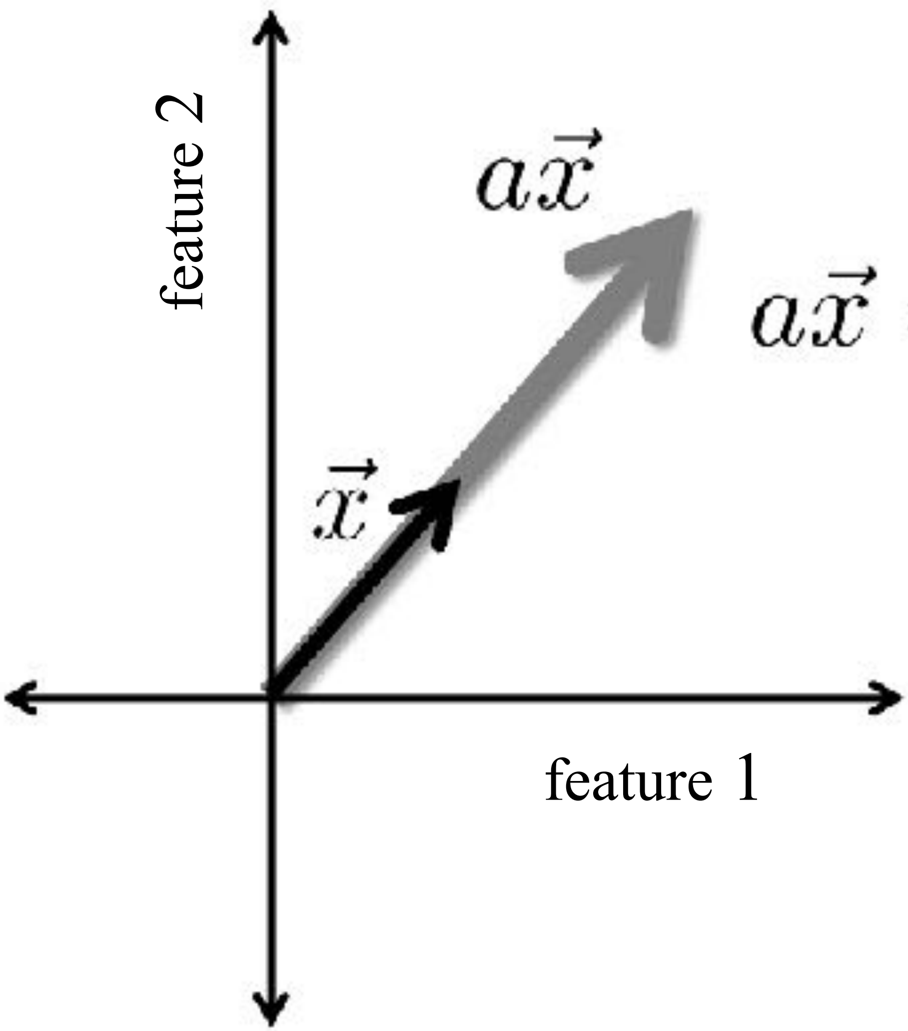
$$\begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1.5 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 0.5 & -0.5 \end{pmatrix}$$

Scalar times vector



Scalar times vector



$$a\vec{x} = a\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_N \end{pmatrix}$$

Product of two vectors

- Element-by-element
- Inner product
- Outer product

Element-by-element product

(Hadamard product)

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} * \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix}$$

Element-by-element product
(Hadamard product)

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} * \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix}$$

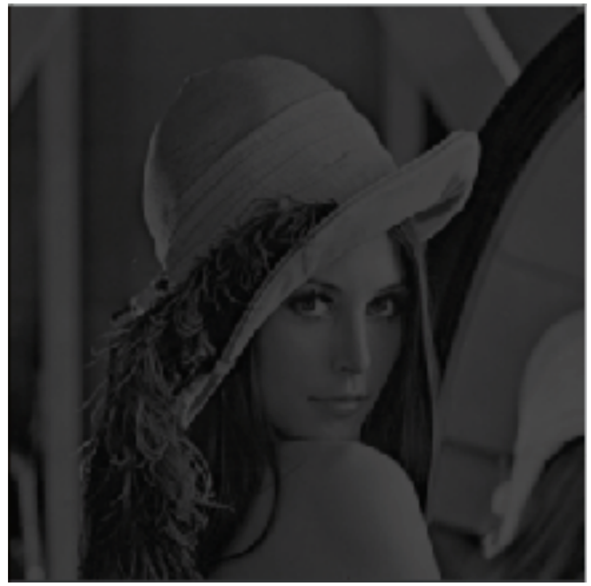
$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} * \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_2 b_2 \\ a_3 b_3 & a_4 b_4 \end{pmatrix}$$

Example

```
[162, 162, 162, ..., 170, 155, 128],
[162, 162, 162, ..., 170, 155, 128],
[162, 162, 162, ..., 170, 155, 128],
...,
[ 43,  43,  50, ..., 104, 100,  98],
[ 44,  44,  55, ..., 104, 105, 108],
[ 44,  44,  55, ..., 104, 105, 108]
```

Example

```
[162, 162, 162, ..., 170, 155, 128],
[162, 162, 162, ..., 170, 155, 128],
[162, 162, 162, ..., 170, 155, 128],
...,
[ 43,  43,  50, ..., 104, 100,  98],
[ 44,  44,  55, ..., 104, 105, 108],
[ 44,  44,  55, ..., 104, 105, 108]
```



$* \; 0.25$

Example

```
[162, 162, 162, ..., 170, 155, 128],
[162, 162, 162, ..., 170, 155, 128],
[162, 162, 162, ..., 170, 155, 128],
...,
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[ 44,  44,  55, ..., 104, 105, 108],
[ 44,  44,  55, ..., 104, 105, 108]
```
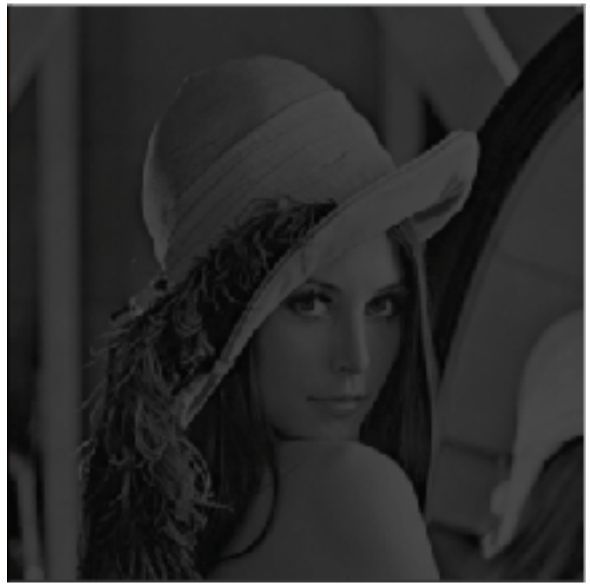
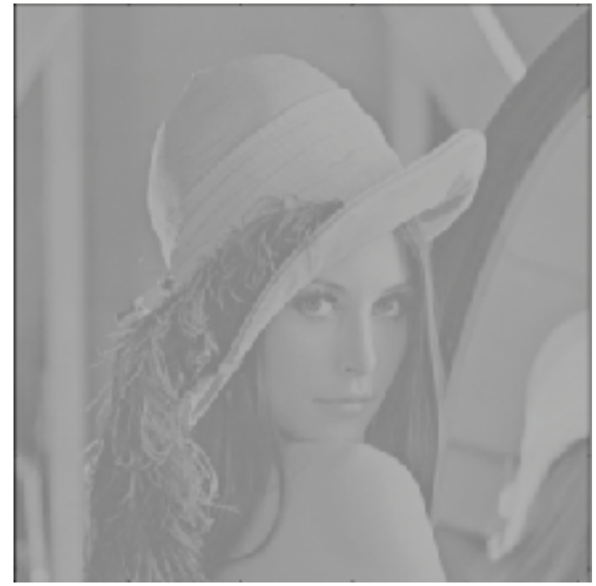
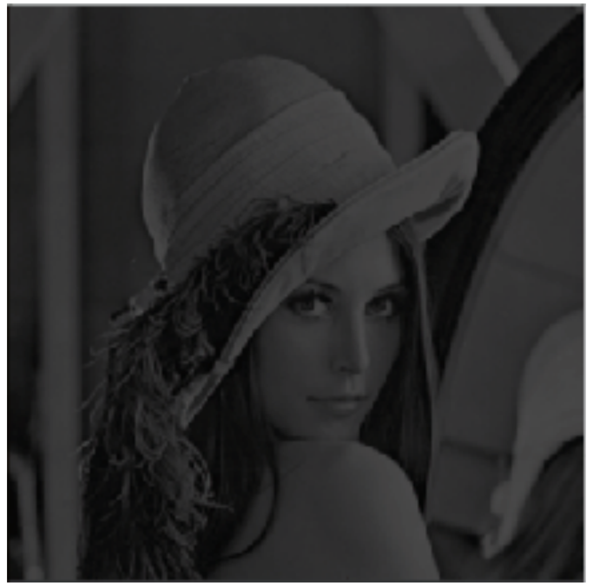



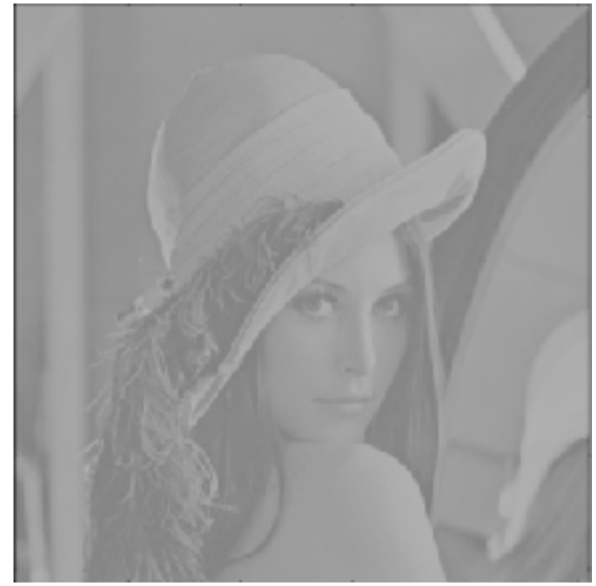
$* \ 0.25$



$+ \ 100$

Example





```
[162, 162, 162, ..., 170, 155, 128],
[162, 162, 162, ..., 170, 155, 128],
[162, 162, 162, ..., 170, 155, 128],
...,
[ 43,  43,  50, ..., 104, 100,  98],
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[ 44,  44,  55, ..., 104, 105, 108]
```

$* \ 0.25$

$+ \ 100$

```
[162, 162, 162, ..., 170, 155, 128],
[162, 162, 162, ..., 170, 155, 128],
[162, 162, 162, ..., 170, 155, 128],
...,
[ 43,  43,  50, ..., 104, 100,  98],
[ 44,  44,  55, ..., 104, 105, 108],
[ 44,  44,  55, ..., 104, 105, 108]
```

*

```
[ 0.25,  0.25, ...,  0.25,  0.25],
[ 0.25,  0.25, ...,  0.25,  0.25],
[ 0.25,  0.25, ...,  0.25,  0.25],
...,
[ 0.25,  0.25, ...,  0.25,  0.25],
[ 0.25,  0.25, ...,  0.25,  0.25],
[ 0.25,  0.25, ...,  0.25,  0.25]
```

+

```
[100, 100, 100, ..., 100, 100, 100],
[100, 100, 100, ..., 100, 100, 100],
[100, 100, 100, ..., 100, 100, 100],
...,
[100, 100, 100, ..., 100, 100, 100],
[100, 100, 100, ..., 100, 100, 100],
[100, 100, 100, ..., 100, 100, 100]
```

Example

(IMG - MIN)*255/MAX

Product of two vectors

- Element-by-element
- Inner product
- Outer product

Multiplication:
Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

Multiplication:

Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1$$

Multiplication:
Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2$$

Multiplication:

Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$$

Multiplication:

Dot product (inner product)

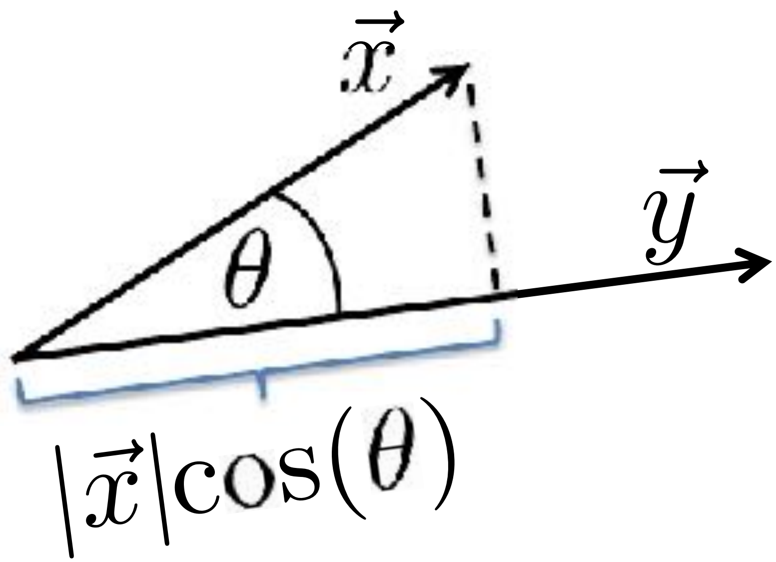
$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$$

$$= \sum_{i=1}^{N} x_i y_i$$

Dot product geometric intuition:

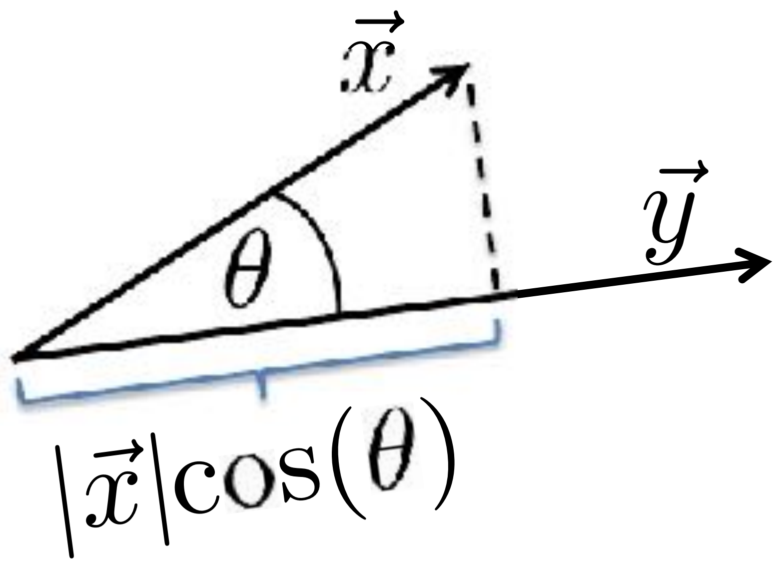
"Overlap" of 2 vectors



$$\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}| \cos(\theta)$$

Dot product geometric intuition:

"Overlap" of 2 vectors

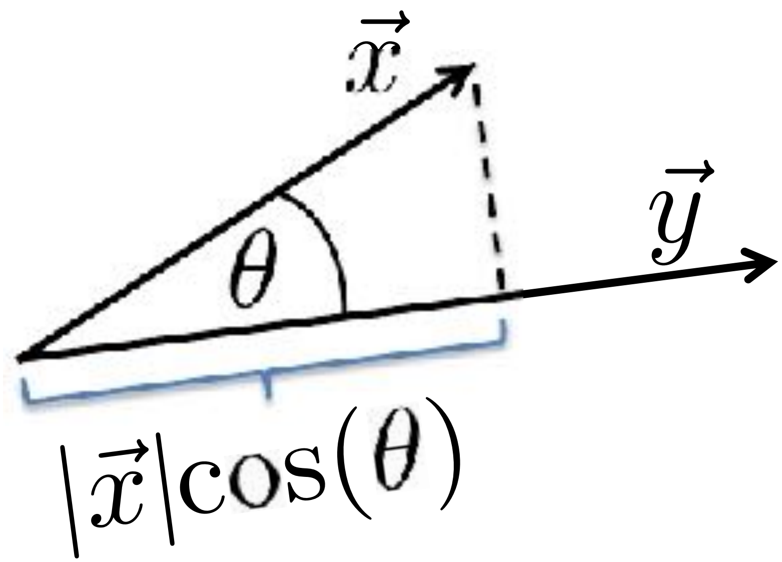


Norm:

$$\vec{x} \cdot \vec{x} = |\vec{x}|^2$$

Dot product geometric intuition:

“Overlap” of 2 vectors



Orthogonal vectors:

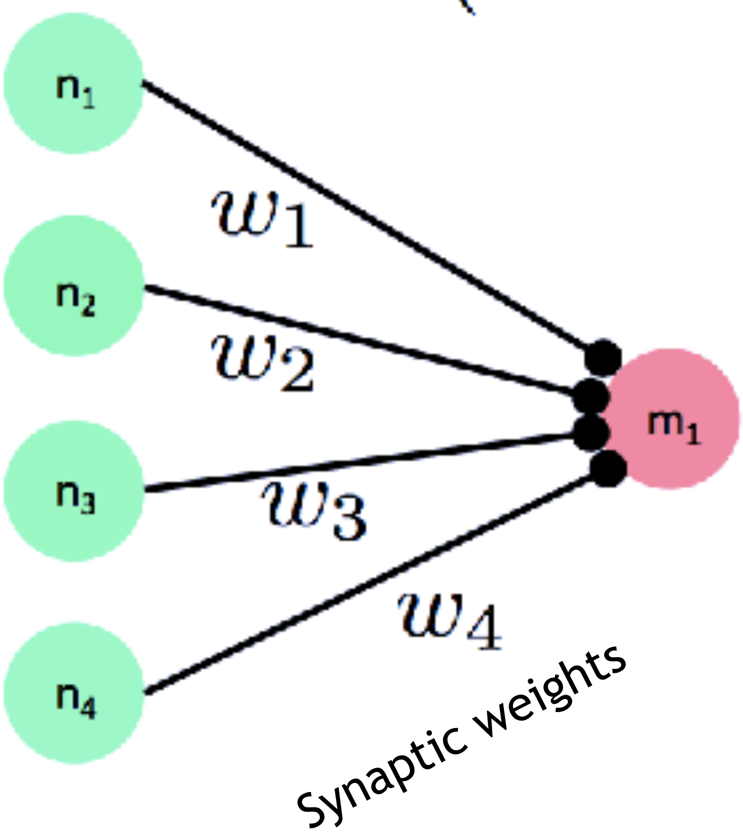
$$\vec{x} \cdot \vec{y} = 0$$

Multiplication:

Dot product (inner product) Example 1

weighted average

$$m_1 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}$$



$$= w_1 n_1 + w_2 n_2 + w_3 n_3 + w_4 n_4$$

$$= w^T n$$

Multiplication:
Dot product (inner product) Example 2

linear regression

patient →

gene 1	1.63	5.20	7.66	8.12	3.22
gene 2	4.98	5.90	8.21	9.29	20.10
gene 3	10.10	8.57	5.73	8.17	2.22
gene 4	0.02	0.21	0.14	0.93	1.40
gene 5	9.27	10.27	13.12	8.90	9.01
gene 6	7.44	6.98	5.62	8.20	7.21

Multiplication:

Dot product (inner product) Example 2

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For each patient, you also measure their Asperger's disorder quotient

Multiplication:
Dot product (inner product) Example 2

linear regression

	Definitely agree	Slightly agree	Slightly disagree	Definitely disagree
1. I prefer to do things with others rather than on my own.	○	○	○	○
2. I prefer to do things the same way over and over again.	○	○	○	○
3. If I try to imagine something, I find it very easy to create a picture in my mind.	○	○	○	○
4. I frequently get so strongly absorbed in one thing that I lose sight of other things.	○	○	○	○
5. I often notice small sounds when others do not.	○	○	○	○
6. I usually notice car number plates or similar strings of information.	○	○	○	○
7. Other people frequently tell me that what I've said is impolite, even though I think it is polite.	○	○	○	○

https://psychology-tools.com/autism-spectrum-quotient/

For each patient, you also measure their Asperger's disorder quotient

Multiplication:

Dot product (inner product) Example 2

linear regression

patient →

gene 1	1.63	5.20	7.66	8.12	3.22
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gene 6	7.44	6.98	5.62	8.20	7.21
score	2	0	9	44	48

Multiplication:
Dot product (inner product) Example 2

linear regression

	patient →				
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gene 2	4.98	5.90	8.21	9.29	20.10
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score	2	0	9	44	48

$$\text{score} = w_1 \text{gene}_1 + w_2 \text{gene}_2 + \cdots + w_6 \text{gene}_6$$

Multiplication:

Dot product (inner product) Example 2

linear regression

$$
\begin{pmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{pmatrix}
\begin{pmatrix} 8.12 \\ 9.29 \\ 8.17 \\ 0.93 \\ 8.90 \\ 8.20 \end{pmatrix}
\begin{array}{l} \text{gene 1} \\ \text{gene 2} \\ \text{gene 3} \\ \text{gene 4} \\ \text{gene 5} \\ \text{gene 6} \end{array}
$$

score 44

$$44 = w_1 8.12 + w_2 9.29 + \cdots + w_6 8.20$$

Multiplication:

Dot product (inner product) Example 2

linear regression

patient →

gene 1	1.63	5.20	7.66	8.12	3.22
gene 2	4.98	5.90	8.21	9.29	20.10
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score	2	0	9	44	48

$$\text{score} = w^T \text{ genes}$$

Product of two vectors

- Element-by-element
- Inner product
- Outer product

Multiplication: **Outer product**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix}$$

N X 1 1 X M

Multiplication: **Outer product**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_M \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_M \\ \vdots & \vdots & \ddots & \cdots \\ x_N y_1 & x_N y_2 & \cdots & x_N y_M \end{pmatrix}$$

N X 1 1 X M N X M

Multiplication: **Outer product**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_M \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_M \\ \vdots & \vdots & \ddots & \ldots \\ x_N y_1 & x_N y_2 & \cdots & x_N y_M \end{pmatrix}$$

Multiplication: **Outer product**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_M \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_M \\ \vdots & \vdots & \ddots & \cdots \\ x_N y_1 & x_N y_2 & \cdots & x_N y_M \end{pmatrix}$$

Multiplication: **Outer product**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_M \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_M \\ \vdots & \vdots & \ddots & \cdots \\ x_N y_1 & x_N y_2 & \cdots & x_N y_M \end{pmatrix}$$

Multiplication: **Outer product**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_M \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_M \\ \vdots & \vdots & \ddots & \cdots \\ x_N y_1 & x_N y_2 & \cdots & x_N y_M \end{pmatrix}$$

Multiplication: **Outer product**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix} = \begin{pmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_M \\ x_2y_1 & x_2y_2 & \cdots & x_2y_M \\ \vdots & \vdots & \ddots & \vdots \\ x_Ny_1 & x_Ny_2 & \cdots & x_Ny_M \end{pmatrix}$$

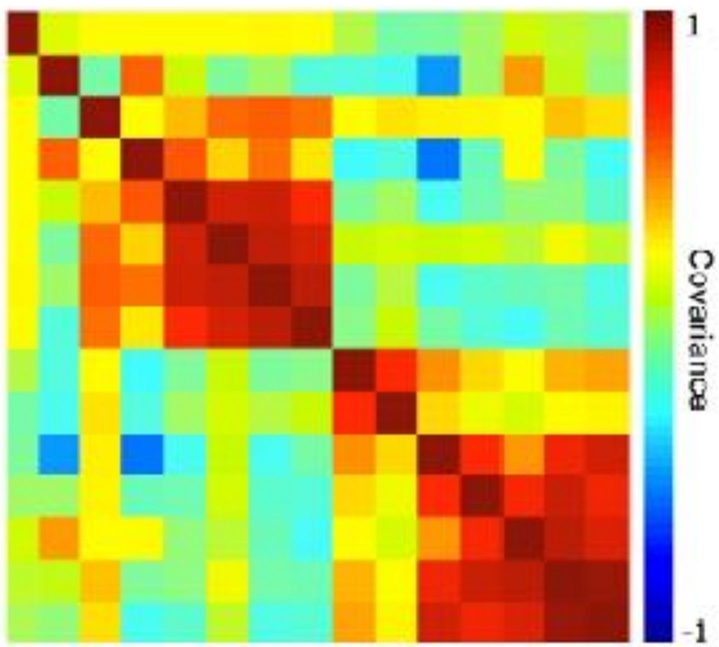
Multiplication: **Outer product**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_M \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_M \\ \vdots & \vdots & \ddots & \vdots \\ x_N y_1 & x_N y_2 & \cdots & x_N y_M \end{pmatrix}$$

- Note: each column or each row is a multiple of the others

Multiplication: **Outer product**

Example: Covariance Matrices



$$= \begin{pmatrix} x_1 x_1 & x_1 x_2 & \ldots x_1 x_N \\ x_2 x_1 & x_2 x_2 & \\ \vdots & & \ddots \\ x_N x_1 & & x_N x_N \end{pmatrix}$$

When $\vec{x} = \vec{y}$ and $\vec{x}$ has an average of zero, this outer product is called the covariance matrix

Matrix times vector

$$\overrightarrow{y} = \overleftrightarrow{W}\,\overrightarrow{x}$$

Matrix times vector

$$\overrightarrow{y} = \overleftrightarrow{W}\,\overrightarrow{x}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

M X 1 M X N N X 1

Matrix times vector:
inner product interpretation

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{i1} & W_{i2} & \cdots & W_{iN} \\ \vdots & \vdots & \ddots & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

- Rule: the i_{th} element of **y** is the dot product of the i_{th} row of W with **x**

Matrix times vector:
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$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{i1} & W_{i2} & \cdots & W_{iN} \\ \vdots & \vdots & \ddots & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

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Matrix times vector: **inner product interpretation**

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{i1} & W_{i2} & \cdots & W_{iN} \\ \vdots & \vdots & \ddots & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

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- Rule: the i_{th} element of **y** is the dot product of the i_{th} row of W with **x**

Matrix times vector:
outer product interpretation

$$
\begin{array}{c}
\begin{array}{cccc} \vec{W}^{(1)} & & & \\ \downarrow & & & \end{array} \\
\begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix}
\end{array}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} =
$$

- The product is a weighted sum of the columns of W, weighted by the entries of x

Matrix times vector: **outer product interpretation**

$$\begin{array}{c} \vec{W}^{(1)} \\ \downarrow \\ \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = x_1 \vec{W}^{(1)} +$$

- The product is a weighted sum of the columns of W, weighted by the entries of x

Matrix times vector:
outer product interpretation

$$
\begin{array}{l}
\begin{array}{c} \vec{W}^{(1)} \\ \downarrow \end{array} \\
\begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = x_1 \vec{W}^{(1)} + x_2 \vec{W}^{(2)}
\end{array}
$$

- The product is a weighted sum of the columns of W, weighted by the entries of x

Matrix times vector:
outer product interpretation

$$\begin{array}{l} \begin{array}{c} \overrightarrow{W}^{(1)} \\ \downarrow \end{array} \\ \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = x_1 \overrightarrow{W}^{(1)} + x_2 \overrightarrow{W}^{(2)} + \cdots + x_N \overrightarrow{W}^{(N)}$$

- The product is a weighted sum of the columns of W, weighted by the entries of x

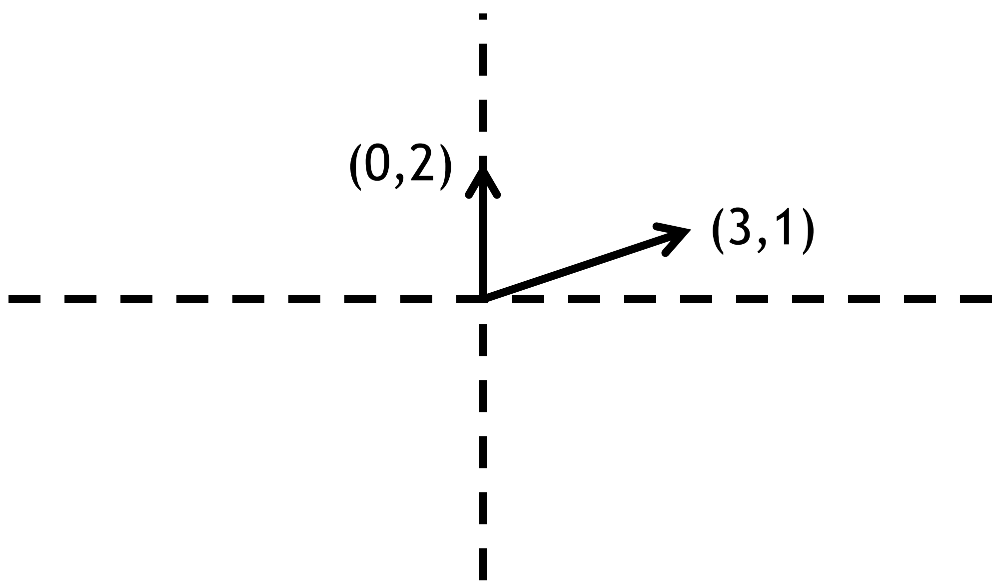
Example of the outer product method

$$\overset{\overleftrightarrow{M}}{\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$

Example of the outer product method

$$\overleftrightarrow{M}$$

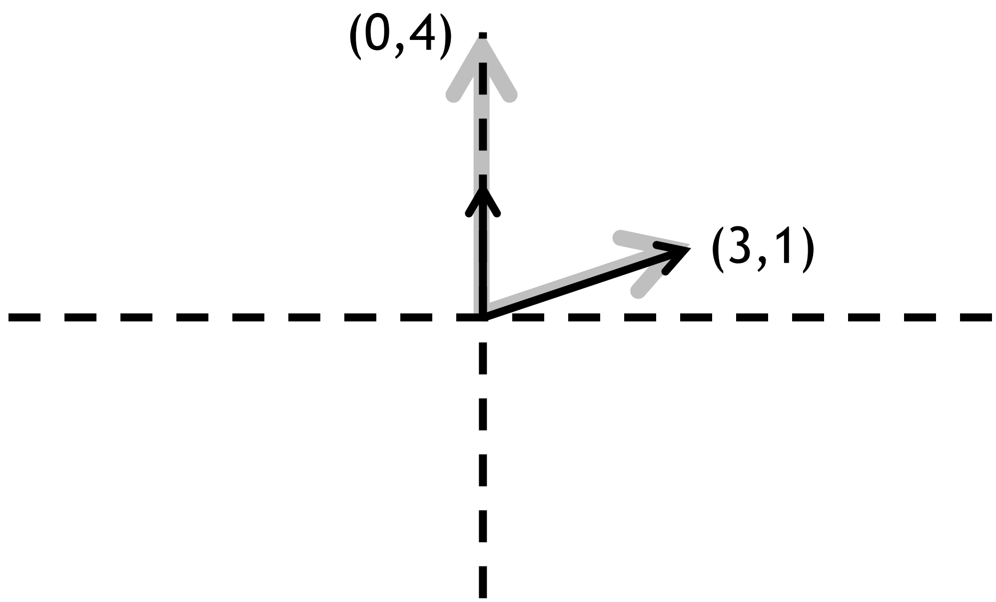
$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



Example of the outer product method

$$\overleftrightarrow{M}$$

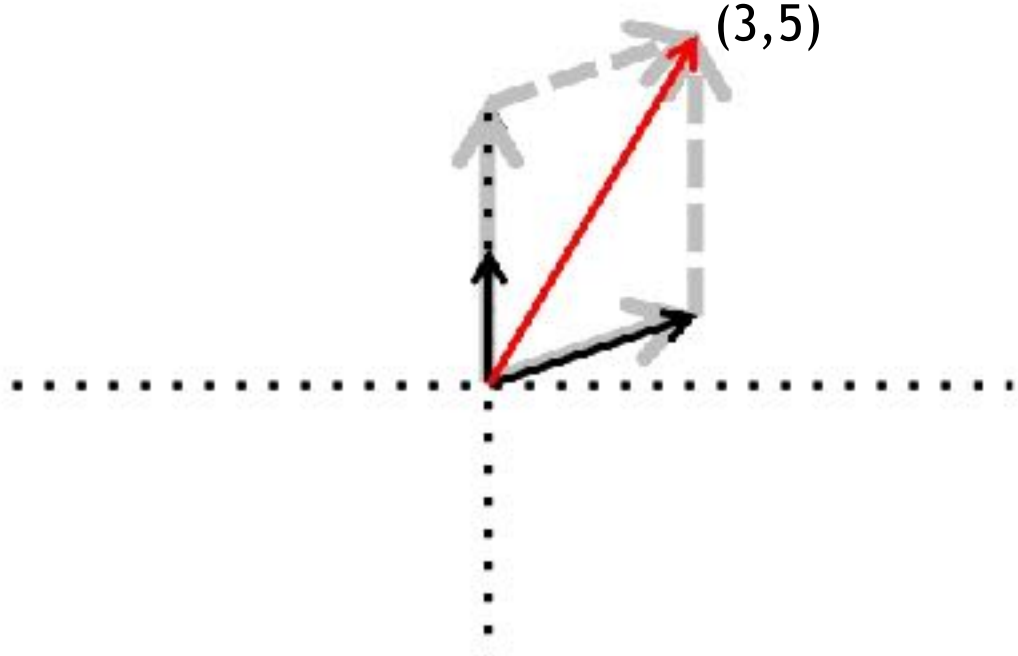
$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



Example of the outer product method

$\overleftrightarrow{M}$

$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$



- Note: different combinations of the columns of **M** can give you any vector in the plane

(we say the columns of **M** "span" the plane)

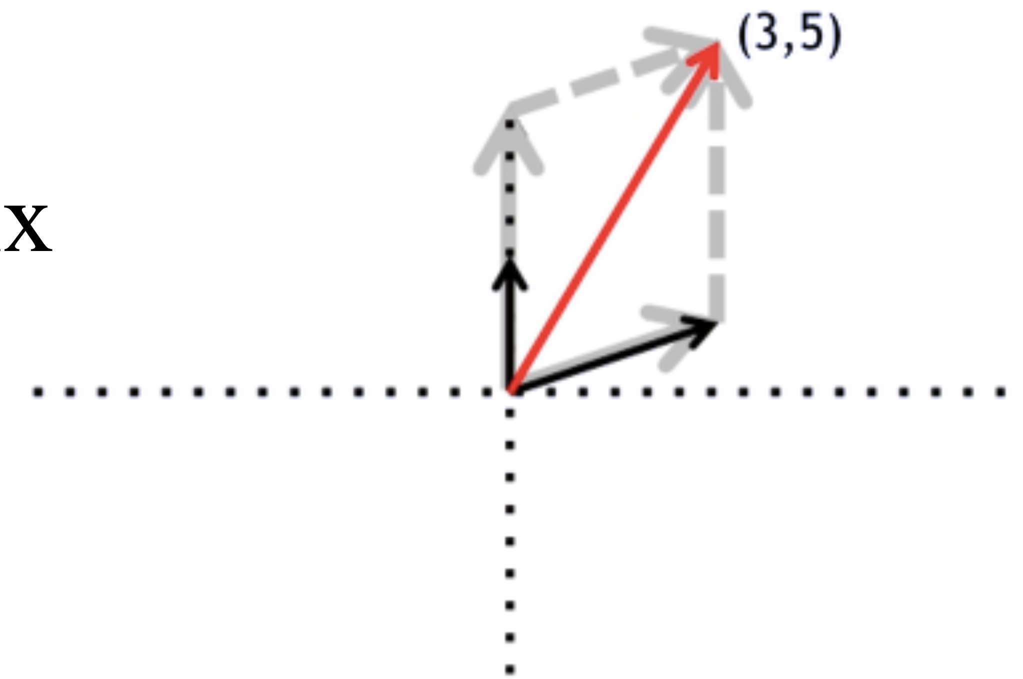
Vector space

- A set of “vectors" with rules for vector addition and multiplication by real numbers
- Conditions: VS includes an identity vector and zero vector, closed under addition and multiplication etc. etc.

Vector subspace

Subset of a vector space, closed under addition and multiplication (should contain zero)

Vector subspace «spanned» by a matrix



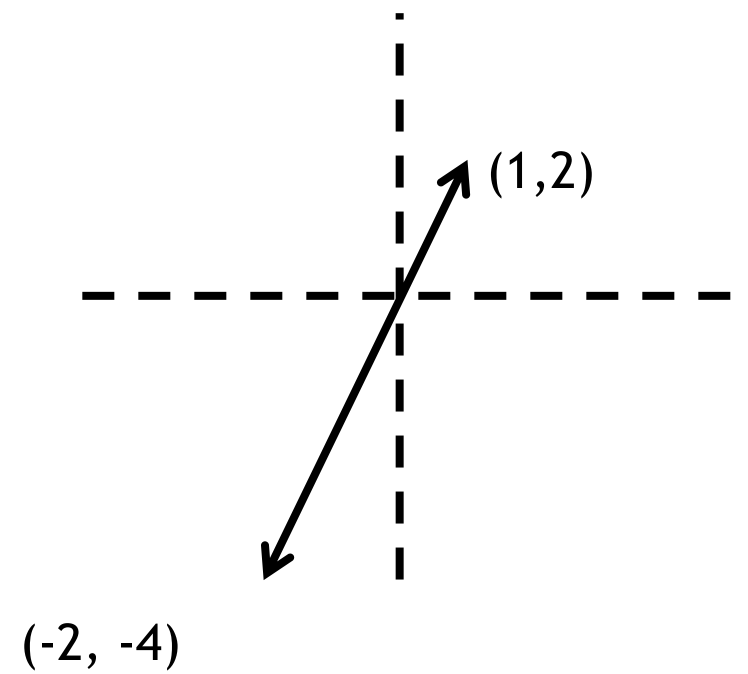
Rank of a Matrix

- Are there special matrices whose columns don’t span the full plane?

Rank of a Matrix

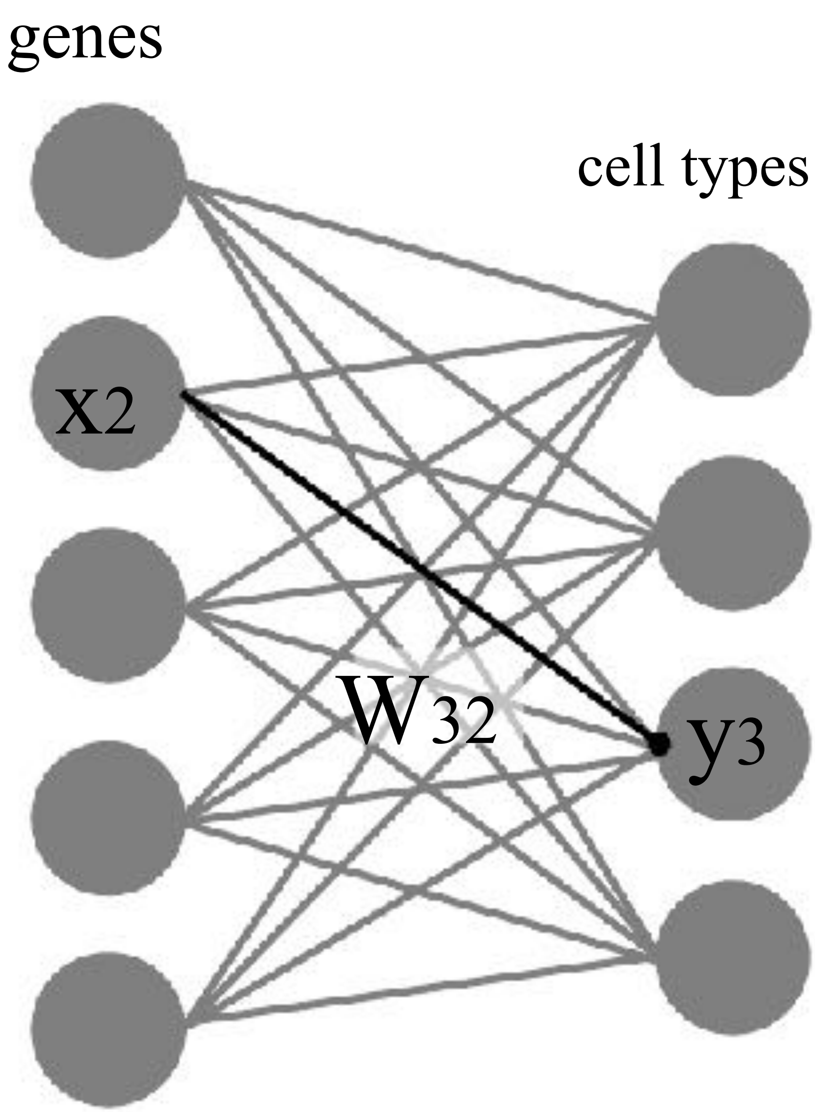
- Are there special matrices whose columns don't span the full plane?

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$$



You can only get vectors along the (1,2) direction (i.e. outputs live in 1 dimension, so we call the matrix *rank 1*)

Example: Development of cell types

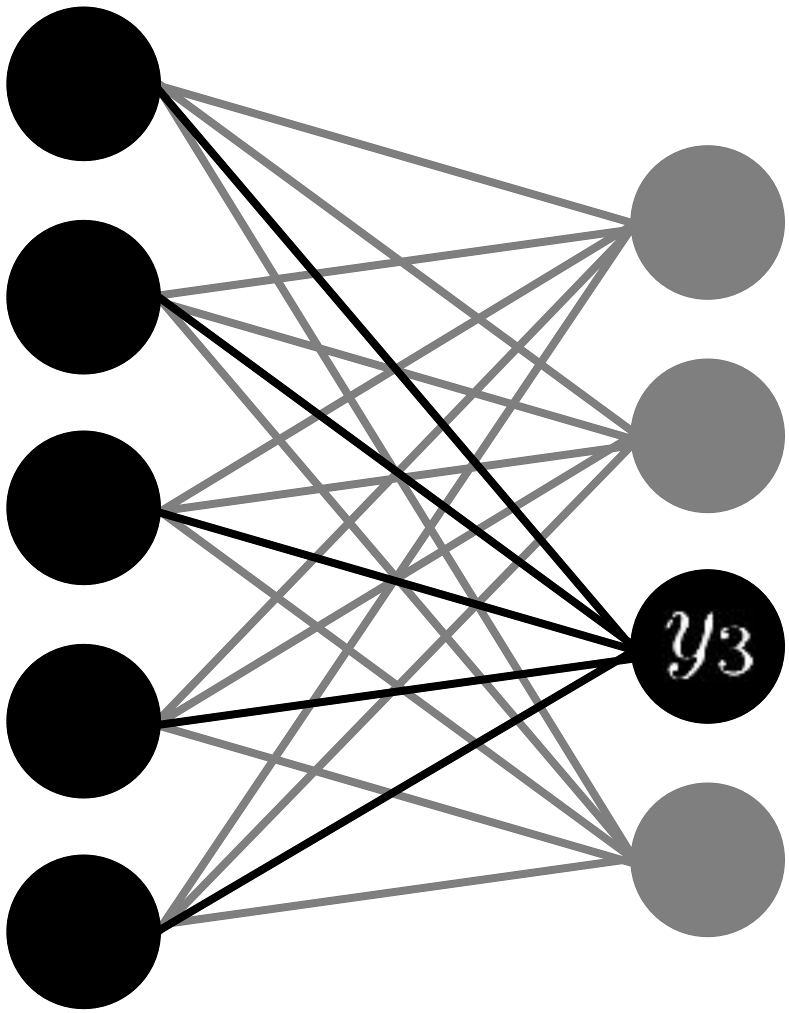


$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

- W_{32} is the influence of gene 2 on developing cell type 3

Example: Development of cell types **inner product point of view**

- *How many cells of type 3 will be created?*

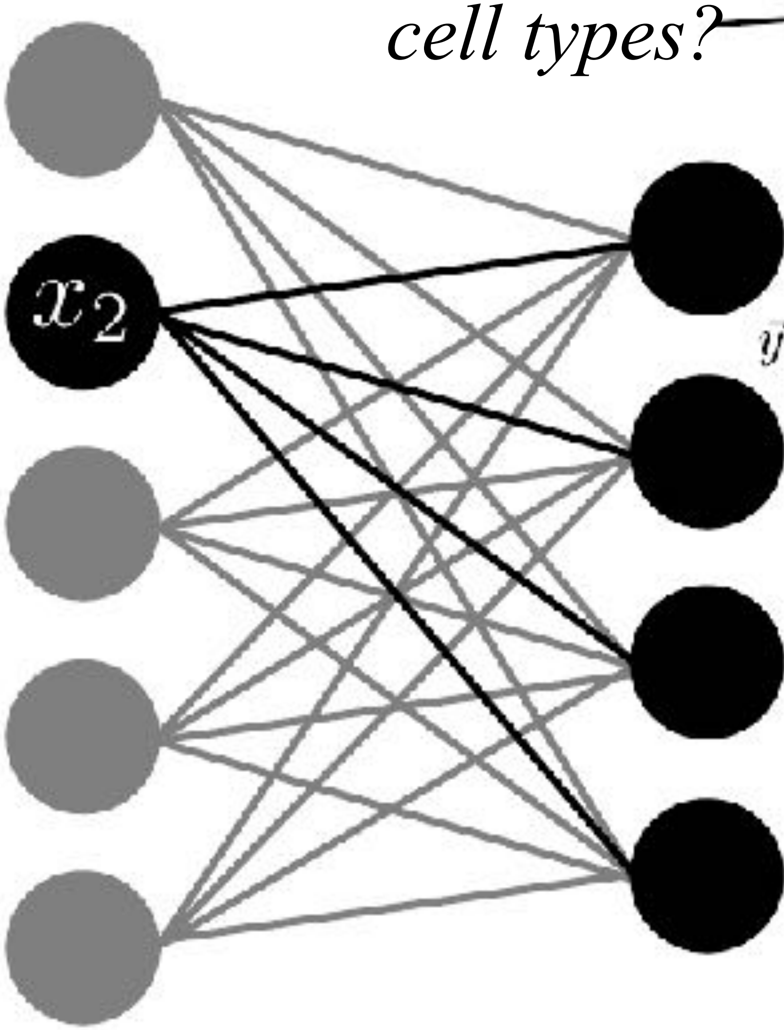


$$y_3 = \sum_{j=1}^{5} W_{3j} x_j$$

- The response is the dot product of the 3rd row of W with the vector x (gene expressions)

Example: Development of cell types: **outer product point of view**

- *How does gene 2 contribute to the distribution of cell types?*



$$\vec{y} = x_1 \overbrace{\overrightarrow{W}^{(1)}}^{\text{1st column of W}} + x_2 \overrightarrow{W}^{(2)} + \cdots + \underbrace{x_j \overrightarrow{W}^{(j)}}_{\text{Contribution of } x_j \text{ to cell types}} + \cdots + x_N \overrightarrow{W}^{(N)}$$

Product of 2 Matrices

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

N X P   P X M   N X M

- inner matrix dimensions must agree
- **Note:** Matrix multiplication doesn't (generally) commute, **AB ≠ BA**

Matrix times Matrix:
by inner products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{iP} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1j} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2j} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{Pj} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & C_{ij} & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

- C_{ij} is the inner product of the i_{th} row of A with the j_{th} column of B

Matrix times Matrix: **by inner products**

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{iP} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1j} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2j} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{Pj} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & C_{ij} & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

- C_{ij} is the inner product of the i_{th} row of A with the j_{th} column of B

Matrix times Matrix:
by outer products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

Matrix times Matrix:
by outer products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

Matrix times Matrix:
by outer products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

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by outer products

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$$\overleftrightarrow{C} = \begin{pmatrix} A^{c1} \end{pmatrix} \begin{pmatrix} B^{r1} \end{pmatrix} + \begin{pmatrix} A^{c2} \end{pmatrix} \begin{pmatrix} B^{r2} \end{pmatrix} + \cdots + \begin{pmatrix} A^{cP} \end{pmatrix} \begin{pmatrix} B^{rP} \end{pmatrix}$$

- **C** is a sum of outer products of the columns of **A** with the rows of **B**

Matrix Properties

- (A few) special matrices
- The determinant
- Eigenvalues and eigenvectors

Special matrices: **diagonal matrix**

$$\overleftrightarrow{D} = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

$$\overleftrightarrow{D}\,\overrightarrow{x} = \begin{pmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{pmatrix}$$

- This acts like scalar multiplication

Special matrices: **identity matrix**

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Special matrices: **inverse matrix**

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Does the inverse always exist?

If a matrix does not have an inverse, it is called singular

Special matrices: **transpose matrix**

$$\mathbf{A}^T$$

- write the rows of **A** as the columns

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

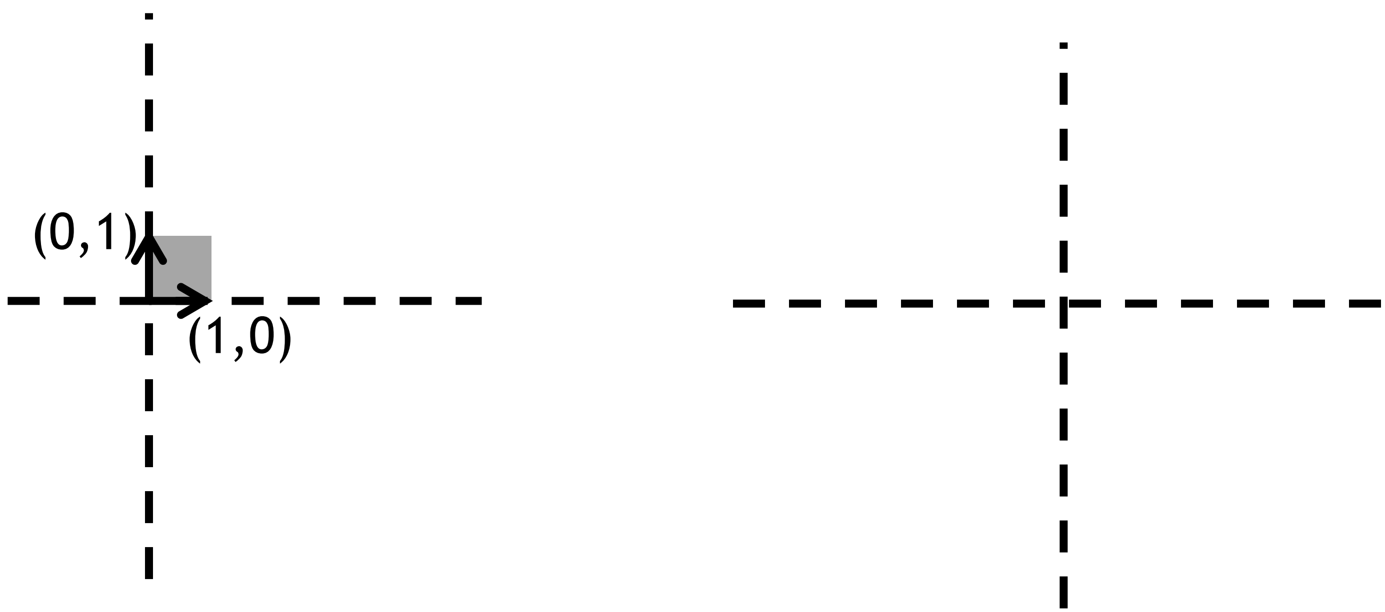
Special matrices: **symmetric matrix**

$$A = A^T$$

Matrix Properties

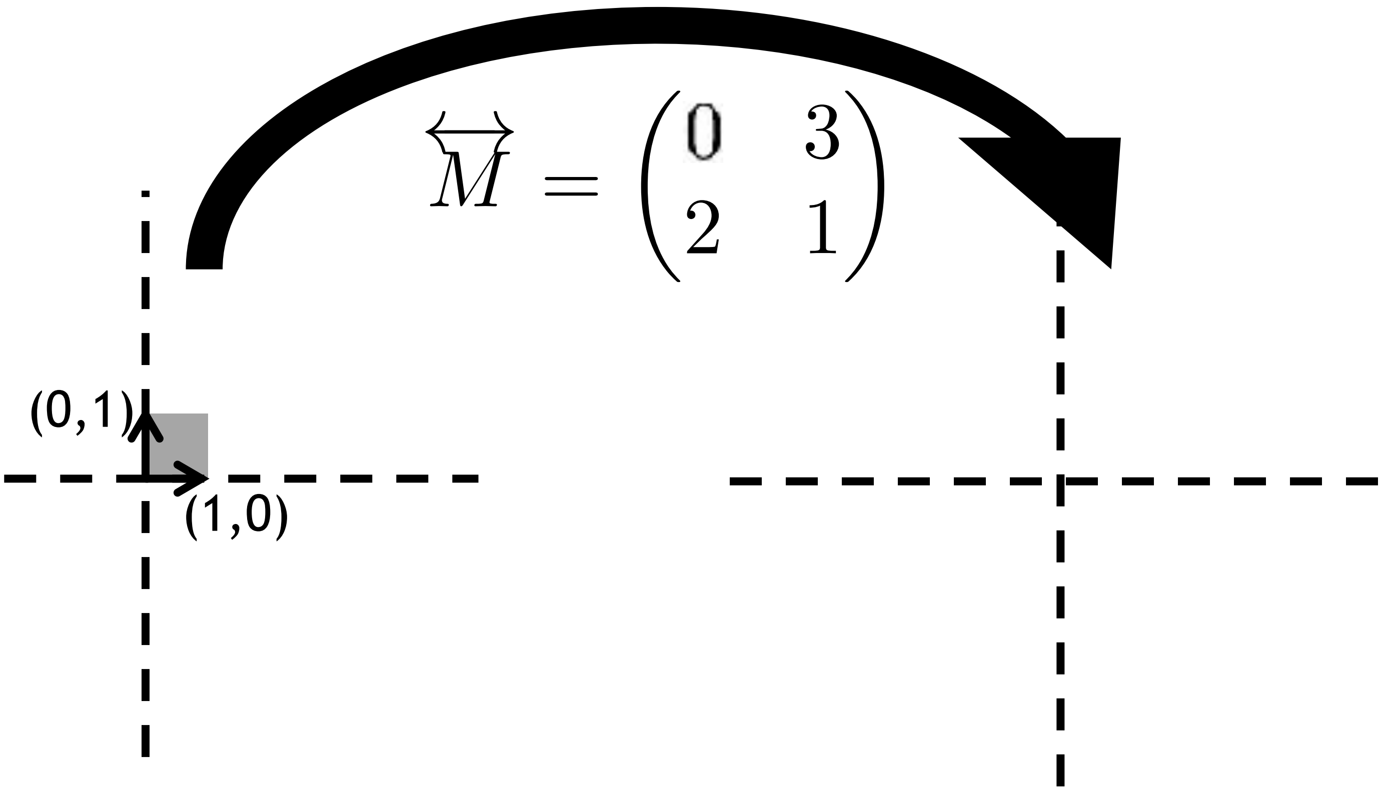
- (A few) special matrices
- The determinant
- Eigenvalues and eigenvectors

How does a matrix transform a square?



How does a matrix transform a square?

$$\overleftrightarrow{M} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$

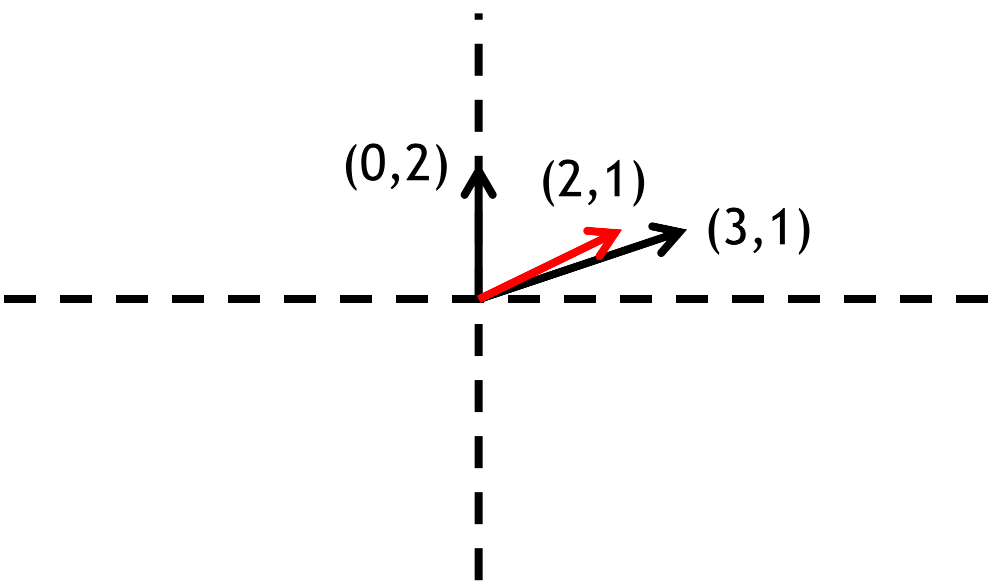


Mx?

What do matrices do to vectors?

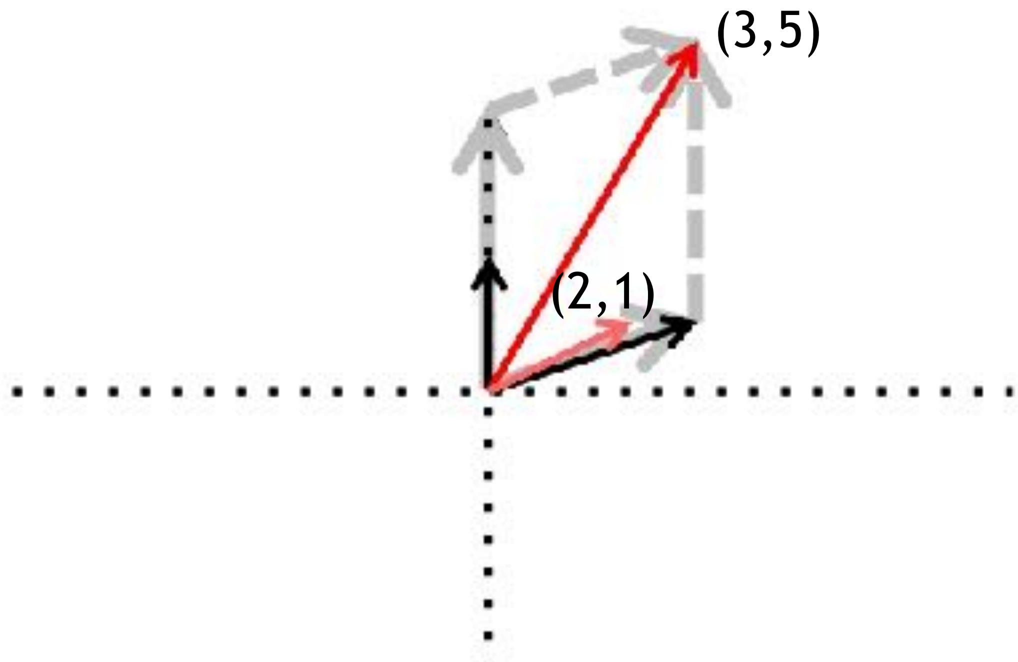
$\overleftrightarrow{M}$

$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$



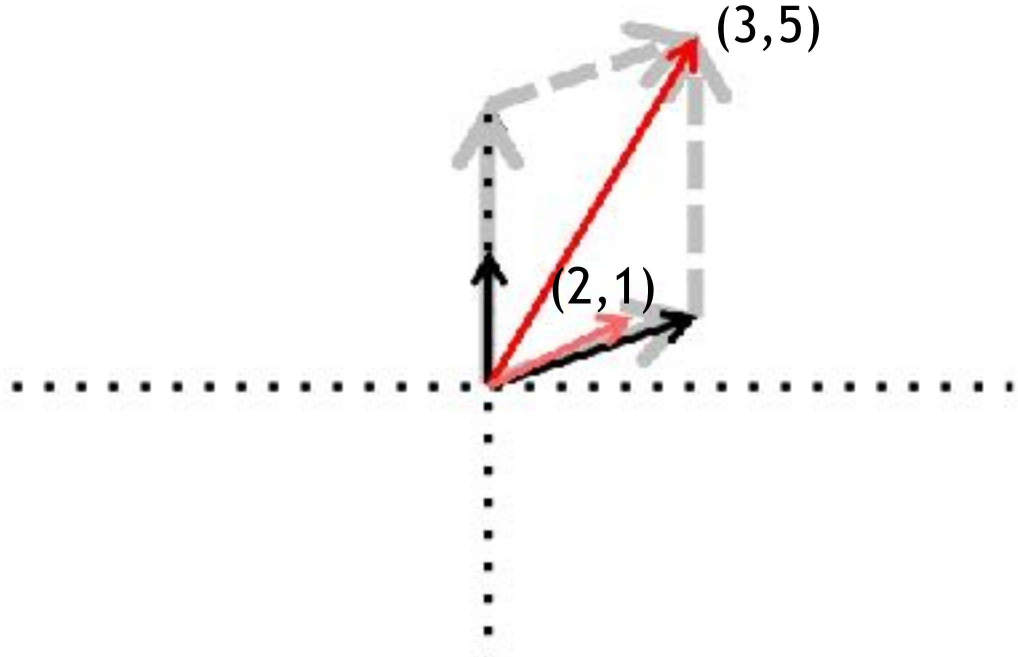
Recall

$$\overbrace{\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}}^{\overleftrightarrow{M}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$



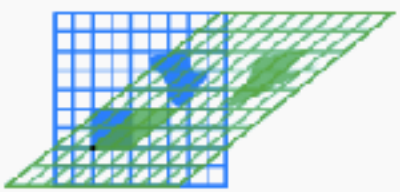
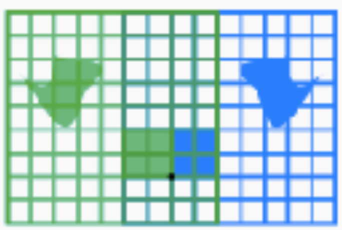
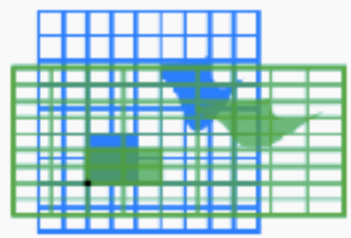
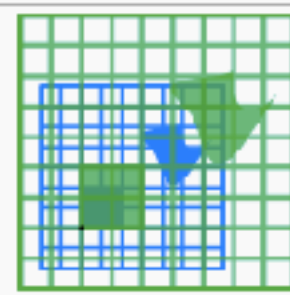
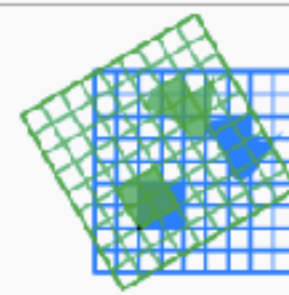
What do matrices do to vectors?

$$\overset{\overleftrightarrow{M}}{\begin{pmatrix}0 & 3\\ 2 & 1\end{pmatrix}}\begin{pmatrix}2\\1\end{pmatrix} = 2\begin{pmatrix}0\\2\end{pmatrix} + 1\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}0\\4\end{pmatrix} + \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix}$$



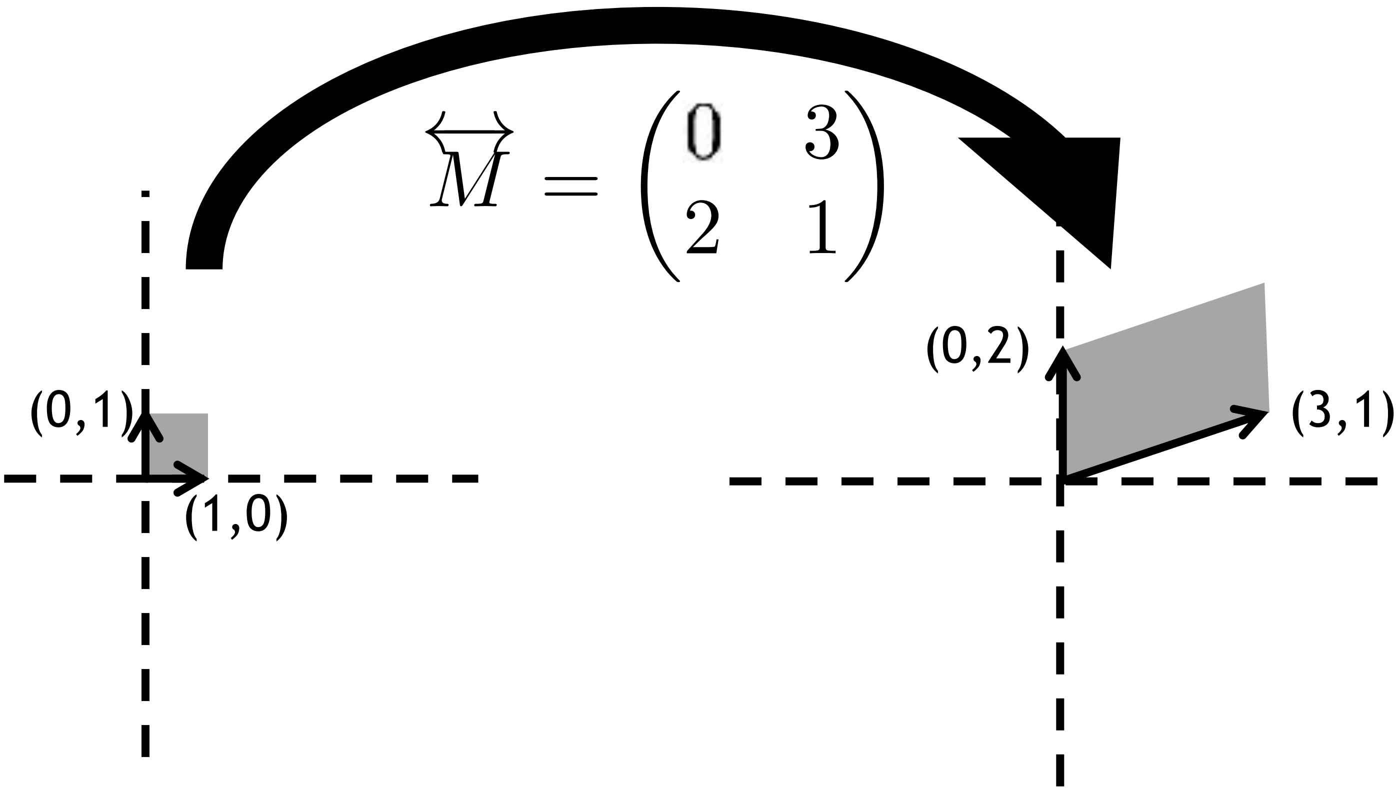
- The new vector is:
  1) **rotated**
  2) **scaled**

What do matrices do to vectors?

Horizontal shear with m=1.25.	Reflection through the vertical axis	Squeeze mapping with r=3/2	Scaling by a factor of 3/2	Rotation by $\pi/6^R = 30°$
$\begin{bmatrix} 1 & 1.25 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 3/2 & 0 \\ 0 & 2/3 \end{bmatrix}$	$\begin{bmatrix} 3/2 & 0 \\ 0 & 3/2 \end{bmatrix}$	$\begin{bmatrix} \cos(\pi/6^R) & -\sin(\pi/6^R) \\ \sin(\pi/6^R) & \cos(\pi/6^R) \end{bmatrix}$
				

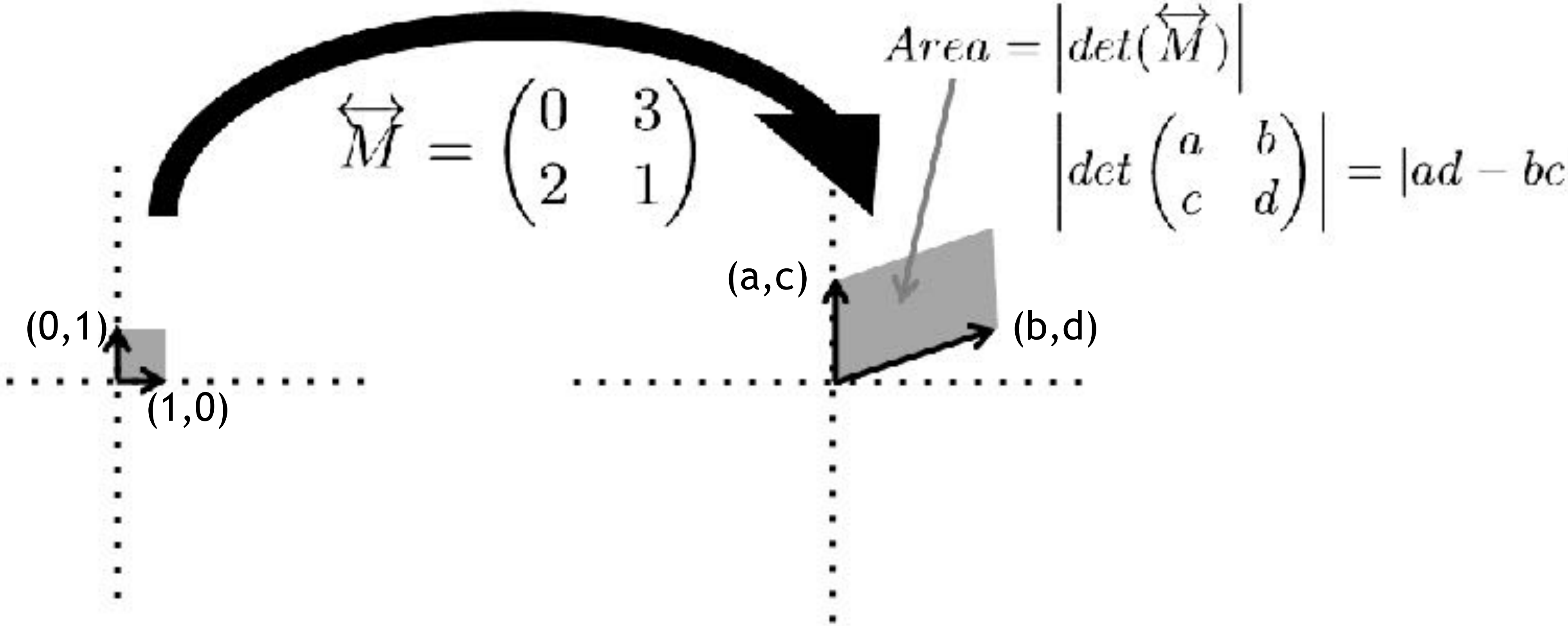
https://en.wikipedia.org/wiki/Matrix_(mathematics)

How does a matrix transform a square?

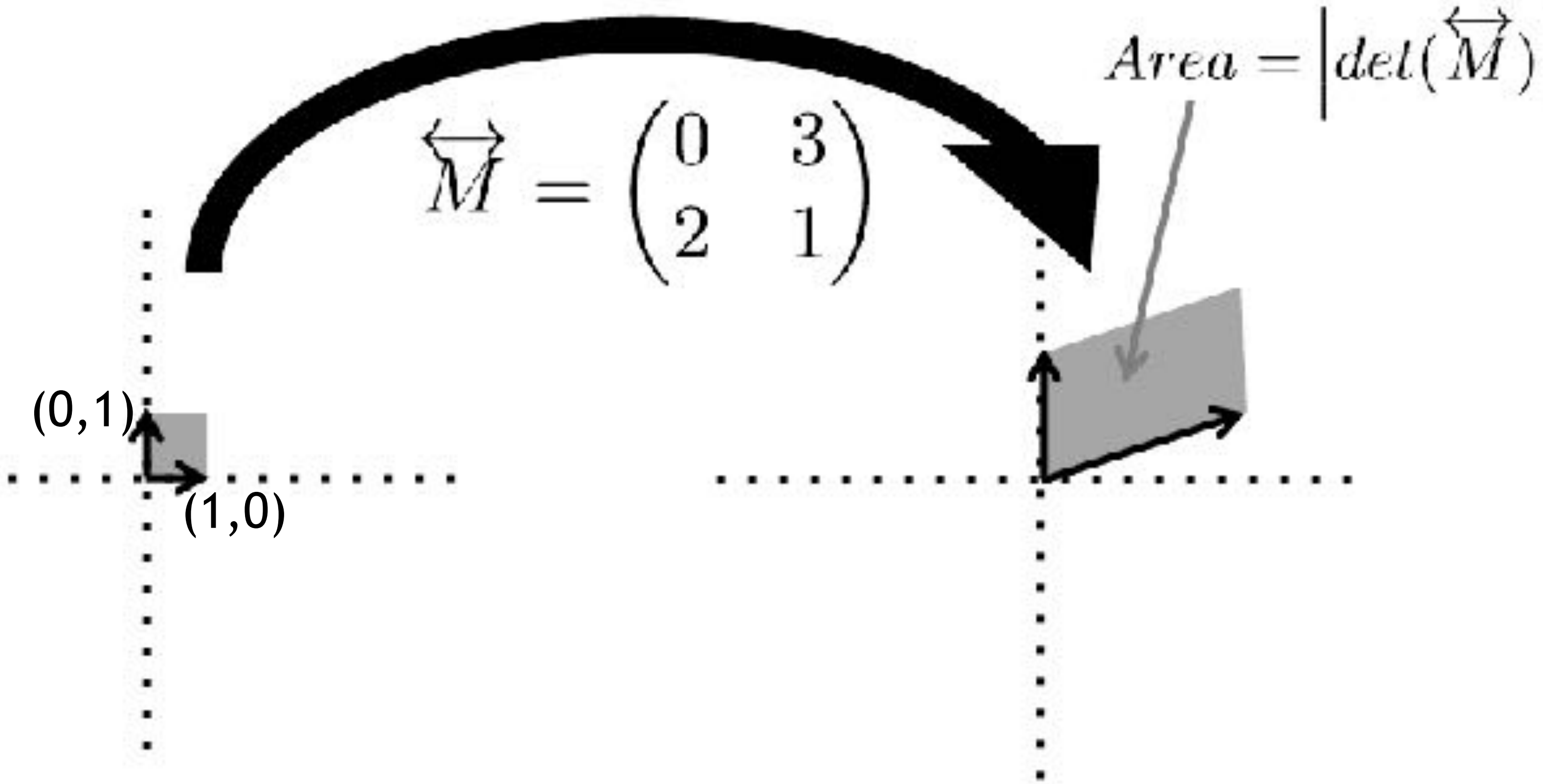


Mx?

Geometric definition of the determinant: How does a matrix transform a square?

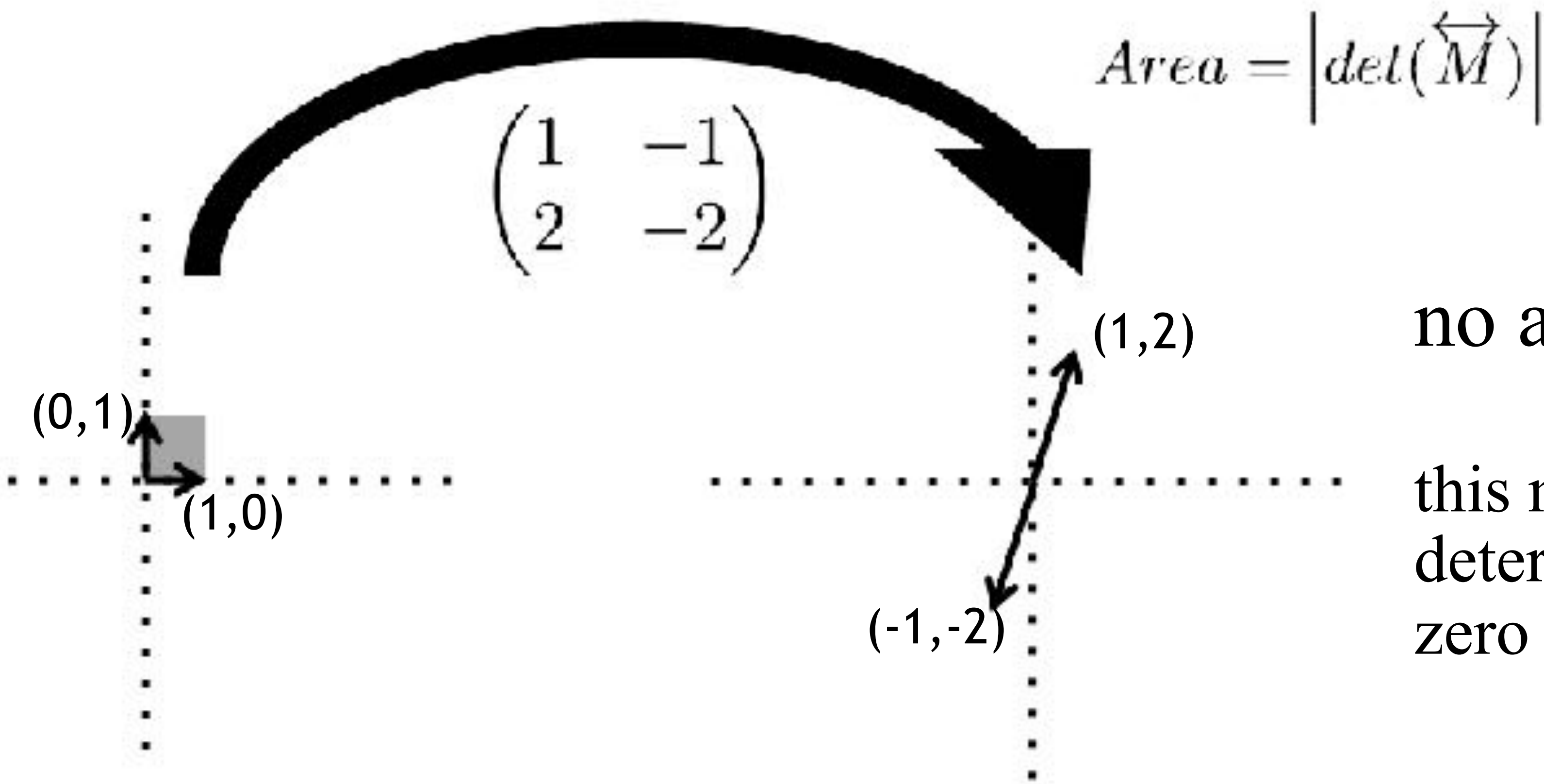


Geometric definition of the determinant: How does a matrix transform a square?



what about $\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$?

Geometric definition of the determinant: How does a matrix transform a square?



no area!

this means the determinant is zero

what about $\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Determinant rules:

$$\begin{aligned} |\mathbf{A}^T| &= |\mathbf{A}| \\ |\mathbf{A}\mathbf{B}| &= |\mathbf{A}||\mathbf{B}| \\ |\mathbf{A}^{-1}| &= 1/|\mathbf{A}|. \end{aligned}$$

Trace of the matrix: sum of diagonal values

$$
\begin{aligned}
\mathrm{tr}[\mathbf{A}^{\mathbf{T}}] &= \mathrm{tr}[\mathbf{A}] \\
\mathrm{tr}[\mathbf{AB}] &= \mathrm{tr}[\mathbf{BA}] \\
\mathrm{tr}[\mathbf{A}+\mathbf{B}] &= \mathrm{tr}[\mathbf{A}] + \mathrm{tr}[\mathbf{B}] \\
\mathrm{tr}[\mathbf{ABC}] &= \mathrm{tr}[\mathbf{BCA}] = \mathrm{tr}[\mathbf{CAB}]
\end{aligned}
$$

$$\mathrm{tr}[\mathbf{ABC}] \neq \mathrm{tr}[\mathbf{BAC}]$$

Diagonal matrix: $\det(\mathbf{A}) = \mathrm{tr}[\mathbf{A}]$

Identity matrix: $\det(\mathbf{I}) = \mathrm{tr}[\mathbf{I}] = 1$

Matrix Properties

- (A few) special matrices
- The determinant
- Eigenvalues and eigenvectors

Eigenvalues and eigenvectors

Let **A** be a squared matrix
λ is an eigenvalue of **A**
if there exists a nonzero vector **x** such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

eigenvalue   eigenvector

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Geometrical interpretation

A defines linear transformation,
x defines a direction
in which deformation is simple stretching / compression

Recall

- Vectors and their operations
  - Element-by-element product
  - Inner product
  - Outer product
- Matrices and their operations
  - Inner product interpretation
  - Outer product interpretation
- Special matrices
- Determinants
- Eigenvalues and eigenvectors